Dual variables for the $SU(2)$ lattice gauge theory at finite temperature

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We study the three-dimensional $SU(2)$ lattice gauge theory at finite temperature using an observable which is dual to the Wilson line. This observable displays a behaviour which is the reverse of that seen for the Wilson line. It is non-zero in the confined phase and becomes zero in the deconfined phase. At large distances, it's correlation function falls off exponentially in the deconfined phase and remains non-zero in the confined phase. The dual variable is non-local and has a string attached to it which creates a $Z(2)$ interface in the system. It's correlation function measures the string tension between oppositely oriented $Z(2)$ domains. The construction of this variable can also be made in the four-dimensional theory where it measures the surface tension between oppositely oriented $Z(2)$ domains.

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Dual variables have played an important role in statistical mechanical systems [1]. These variables display a behaviour which is the opposite of that seen for the order parameters. They are non-zero in the disordered phase and remain zero in the ordered phase. Hence they are commonly referred to as disorder variables. Unlike the order parameters which are local observables and measure long range order in a statistical mechanical system, the dual variables are non-local and are sensitive to disordering effects which often arise as a consequence of topological excitations supported by a system - like vortices, magnetic monopoles etc. Disorder variables for the $U(1)$ LGT have been studied recently [2]. In this paper we study the finite temperature properties of the three-dimensional $SU(2)$ lattice gauge theory using an observable which is dual to the Wilson line. We explain the sense in which this is dual to the Wilson line and show that it's behaviour is the reverse of that observed for the Wilson line. Unlike the Wilson line which creates a static quark propagating in a heat bath, the dual variable creates a $Z(2)$ interface in the system. The definition of this variable can also be extended to the four-dimensional theory.

Before we consider the three-dimensional $SU(2)$ lattice gauge theory let us briefly recall the construction of the dual variable for the two-dimensional Ising model [3]. The variable dual to the spin variable $\sigma(\vec{n})$ is denoted by $\mu(\star\vec{n})$ and is defined on the dual lattice. This variable which is shown in Fig. 1 has a string attached to it which pierces the bonds connecting the spin variables. The position of the string is not fixed and it can be varied using a $Z(2)$ ($\sigma(\vec{n}) \to -\sigma(\vec{n})$) transformation. The average value of the dual variable is defined as

$$< \mu(\star\vec{n}) >= \frac{Z(\tilde{K})}{Z(K)} \tag{1}$$

where $Z(K)$ and $Z(\tilde{K})$ are the partition functions defined using the coupling constants K and $\tilde{K}$ respectively. The partition function for the Ising model is got from the Hamiltonian

$$H = -K \sum_{\vec{n}\vec{n}'} \sigma(\vec{n})\sigma(\vec{n}'). \tag{2}$$

The coupling constant $\tilde{K}$ is defined as

$$\tilde{K} = -K \;\; on\ the\ bond\ pierced\ by\ the\ string$$

$$\tilde{K} = K \;\; elsewhere.$$

The dual variable $\mu(\star\vec{n})$ thus creates an interface beginning from $\star\vec{n}$. It has the following behaviour at high and low temperatures [3]

$$< \mu(\star\vec{n}) > \approx 1 \; for \; K \; small$$

$$< \mu(\star\vec{n}) > \approx 0 \; for \; K \; large.$$

It is in this sense that the variable $\mu(\star\vec{n})$ is dual to the variable $\sigma(\vec{n})$ which behaves as

$$< \sigma(\vec{n}) > \approx 0 \; for \; K \; small$$

$$< \sigma(\vec{n}) > \approx 1 \; for \; K \; large.$$

The spin and dual correlation functions satisfy the relation

$$< \mu(\star\vec{n})\mu(\star\vec{n}') >_{K>>1} = < \sigma(\vec{n})\sigma(\vec{n}') >_{K<<1} . \tag{3}$$

Using the $\sigma \to -\sigma$ transformation it can be shown that the correlation function of the μ's is independent of the shape of the string joining $\star\vec{n}$ and $\star\vec{n}'$. The variables $\sigma(\vec{n})$ and $\mu(\star\vec{n})$ satisfy the algebra

$$\sigma(\vec{n})\mu(\star\vec{n}) = \mu(\star\vec{n})\sigma(\vec{n})\exp(i\omega), \tag{4}$$

where $\omega = 0$ if the variable σ does not lie on a bond pierced by the string attached to $\mu(\star\vec{n})$ and $\omega = \pi$ otherwise.

The above considerations generalize easily to the three-dimensional $Z(2)$ gauge theory. The dual variables are again defined on the sites of the dual lattice and the string attached to them will now pierce plaquettes instead of bonds. Whenever a plaquette is pierced by a string the coupling constant changes sign just as in the case of the Ising model. One can similarly define correlation functions of these variables. Since the three-dimensional $Z(2)$ gauge theory is dual to the the three- dimensional Ising model, the correlation functions of these variables will have a behaviour which is the reverse of the spin-spin correlation function in the three-dimensional Ising model. For the case of the $SU(2)$ lattice gauge theory which is our interest here, the definition of these variables is more involved. However, since $Z(2)$ is a subgroup of $SU(2)$ one can define variables which are dual to the $Z(2)$ degrees of freedom by following the same prescription as in the three-dimensional $Z(2)$ gauge theory. The relevance and effectiveness of these variables will depend

on the role played by the $Z(2)$ degrees of freedom in the $SU(2)$ lattice gauge theory. The role of the center degrees of freedom in the $SU(2)$ lattice gauge theory was also examined in [4].

Since the finite temperature transition in $SU(N)$ lattice gauge theories is governed by the center ($Z(N)$ for $SU(N)$) degrees of freedom [5], we expect these variables to be useful in studying this transition. The usual analysis of finite temperature lattice gauge theories is carried out by studying the behaviour of the Wilson line which becomes non-zero across the finite temperature transition [5]. The non-zero value of the Wilson line indicates deconfinement of static quarks. The spatial degrees of freedom undergo no dramatic change across the transition and only serve to produce short-range interactions between the Wilson lines. Thus one gets an effective theory of Wilson lines in one lower dimension [6]. The deconfinement transition can be monitored by either measuring the expectation value of the Wilson line or by looking at the behaviour of the Wilson line correlation function [7]. In the confining phase, the correlation function is (for $|\vec{n} - \vec{n}'|$ large)

$$< L(\vec{n})L(\vec{n}') >\approx \exp(-\sigma T|\vec{n} - \vec{n}'|) \tag{5}$$

while in the deconfining phase

$$< L(\vec{n})L(\vec{n}') >\approx constant. \tag{6}$$

We define the variable $\mu(\star\vec{n})$ on the dual lattice site $\star\vec{n}$ as

$$\mu(\star\vec{n}) = \frac{Z(\tilde{\beta})}{Z(\beta)} \tag{7}$$

where $Z(\tilde{\beta})$ is the partition function with couplings $\tilde{\beta}$ which is defined as

$$\tilde{\beta} = -\beta \ on \ plaquettes \ pierced \ by \ the \ string$$

$$\tilde{\beta} = \beta \ elsewhere$$

and the string runs in the spatial direction. Hence the plaquettes pierced by the string are all space-time plaquettes. The action for the $SU(2)$ LGT is chosen to be the Wilson action [9] which is

$$S = \frac{\beta}{2} \sum_p tr \ U(p). \tag{8}$$

The variables $\mu(\star\vec{n})$ and $L(\vec{n})$ satisfy the algebra

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$$L(\vec{n})\mu(\star\vec{n}) = \mu(\star\vec{n})L(\vec{n})\exp(i\omega) \qquad (9)$$

where $\omega = 0$ if the plaquette pierced by the string attached to $\mu(\star\vec{n})$ is not touching any of the links belonging to $L(\vec{n})$ and $\omega = \pi$ if the plaquette makes contact with any of the links of $L(\vec{n})$. The variables $\mu(\star\vec{n})$ and $L(\vec{n})$ satisfy the same algebra as the σ and μ variables in the Ising model. This is the same as the algebra of the order and disorder variables in [8]. Note that this algebra is only satisfied if the string is taken to be in the spatial direction. The location of the string can again be changed by local $Z(2)$ transformations. The correlation function of the dual variables is defined to be

$$< \mu(\star\vec{x})\mu(\star\vec{y}) >= \frac{Z(\tilde{\beta})}{Z(\beta)} \qquad (10)$$

where $\tilde{\beta} = -\beta$ on all plaquettes pierced by the string joining $\vec{x}$ and $\vec{y}$ and $\tilde{\beta} = \beta$ otherwise. It is again easily seen that this quantity is independent of the shape of the string which can always be varied by a $Z(2)$ $(U(n;\mu) \to -U(n;\mu))$ transformation. The righthand side of Eq. 10 can be expressed as an average value

$$< \mu(\star\vec{x})\mu(\star\vec{y}) >=< \exp(-\beta\sum_{p'} tr\ U(p)) >_\beta \qquad (11)$$

where the prime denotes that the summation is only over plaquettes which are dual to the string joining $\vec{x}$ and $\vec{y}$, and the average is taken using the coupling β. Since the string is spatial, all the plaquettes appearing in the sum are space-time plaquettes.

It is easy to show that the behaviour of this correlation function will be the reverse of that of the Wilson line. To see this note that when the spatial degrees of freedom are integrated out, we get an effective two- dimensional model of Ising like spins with local interactions. To leading order in strong coupling, the effective action for the Wilson lines has the form

$$S_{eff} = 2(\frac{\beta}{2})^{N_\tau} \sum_{\vec{n}\vec{n}'} J(\vec{n} - \vec{n}')trL(\vec{n})\ trL(\vec{n}'). \qquad (12)$$

The term which gives this contribution is shown in Fig. 2. When we calculate the correlation function in Eq. 10 (where $\vec{x}$ and $\vec{y}$ are only separated in space) using this approximation, one plaquette occurring in this diagram will contribute with the opposite sign (shown shaded in Figure. 2) and will cause the bond between $\vec{n}$ and $\vec{n}'$ to have a coupling with the opposite sign. In Eq. 12 $J(\vec{n} - \vec{n}')$ contains the sign induced

on the bond. This feature will persist for every diagram contributing to the effective two-dimensional Ising model and it's effect will be to create a disorder line from $\vec{x}$ to $\vec{y}$. Thus this correlation function will behave exactly like the disorder variable in the two-dimensional Ising model and at large distances will fall off exponentially in the ordered phase and will approach a constant value in the disordered phase. We expect it to behave (for large $|\vec{x} - \vec{y}|$) as

$$< \mu(\vec{x})\mu(\vec{y}) > \approx exp(-|\vec{x} - \vec{y}|/\xi) \quad \beta > \beta_{cr}$$

$$< \mu(\vec{x})\mu(\vec{y}) > \approx \mu^2 \quad \beta < \beta_{cr}$$

Writing the above correlation function as

$$< \mu(\vec{x})\mu(\vec{y}) > = \exp(-\beta_\tau(F(\vec{x} - \vec{y}))) \tag{13}$$

we can interpret F as the free energy of an interface of length $|\vec{x} - \vec{y}|$. The inverse temperature is denoted by β_τ to distinguish it from the gauge theory coupling β. In the ordered phase the interface energy increases linearly with the length of the interface while in the disordered phase it is independent of the length. In the finite temperature system high temperature results in the ordering of the Wilson lines and low temperature results in the disordering of the Wilson lines. Therefore the dual variables will display ordering at low temperatures and disordering at high temperatures.

A direct measurement of the dual variable results in large errors because the dual variable is the exponential of a sum of plaquettes and fluctuates greatly. We have directly measured the dual variable and the correlation function and found that they fall to zero at high temperatures and remain non-zero at low temperatures. Since the measurement had large errors we prefer to use the method in [11] where a similar problem was encountered in the measurement of the disorder variable in the $U(1)$ LGT. Instead of directly measuring the correlation function we measure

$$\rho(\vec{x}, \vec{y}) = -\frac{\partial ln < \mu >}{\partial \beta}. \tag{14}$$

This quantity can be rewritten as

$$\rho(\vec{x}, \vec{y}) = - < \sum_{p \neq p'} (1/2) tr\ U(p) >_{\tilde{\beta}} + < \sum_{p = p'} (1/2) tr\ U(p) >_{\beta} \tag{15}$$

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where p' denotes the plaquettes which are dual to the string joining $\vec{x}$ and $\vec{y}$. In our case this quantity directly measures the free energy of the $Z(2)$ interface between $\vec{x}$ and $\vec{y}$. Hence we expect it to increase linearly with the interface length in the deconfining phase and approach a constant value in the confining phase. Also this variable is like any other statistical variable and is easier to measure numerically. The variable ρ can be used to directly measure the interface string tension between oppositely oriented $Z(2)$ domains. The behaviour of the quantity ρ is shown in Fig. 3 and Fig. 4. In the confined phase ρ approaches a constant value at large distances while it increases linearly with distance in the deconfined phase. The slope of the straight line in Fig. 3 gives the interface string tension. The calculation of ρ was made on a $12**2\ 3$ lattice with 200000 iterations. The values of β used were 2.5 in the confined phase and 5.5 in the deconfined phase. The deconfinement transition on the $N_\tau = 3$ lattice occurs at $\beta = 4.1$ [10]. The errors were estimated by blocking the data.

We would now like to point out a few applications of these dual variables. The mass gap in the high temperature phase is determined by studying the large distance behaviour of the Wilson line correlation function. Since the Wilson line correlation function remains non-zero in the deconfined phase the long distance part is subtracted out to get the leading exponential. The dual variable correlation function already displays an exponential fall off in the high temperature phase and provides us with another method of estimating the mass gap. Also, since dual variables reverse the roles of strong and weak coupling, they provide an alternate way of looking at the system which may be convenient to address certain questions. In this case they can be used to determine the string tension between oppositely oriented $Z(2)$ domains in the $SU(2)$ gauge theory. The surface tension between oppositely oriented $Z(2)$ domains in the four-dimensional theory has been calculated semi-classically in [12] .

The above construction of the dual variable can also be made in four dimensions. The only difference is that in four dimensions the dual variables are defined on loops in the dual lattice. The spatial string in three-dimensions is replaced by a spatial surface which has the loops as it's the boundary. The dual variables are functionals of the surface bounding the loops. The correlation function of the dual variables is defined to be

$$< \mu(C, C') >=< exp(-\beta \sum_{p'} tr\ U(p)) > \tag{16}$$

where the summation is over all plaquettes which are dual to the surface joining C and C'. Since the surface is purely spatial the plaquettes contributing to the summation are all space-time plaquettes. This correlation function will fall of exponentially as the area of the surface joining C and C' in the deconfined phase and will approach a constant value in the confined phase. A similar measurement of ρ can be used to determine the surface tension between oppositely oriented $Z(2)$ domains in the four-dimensional gauge theory.

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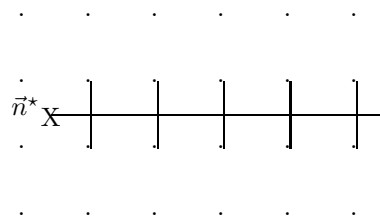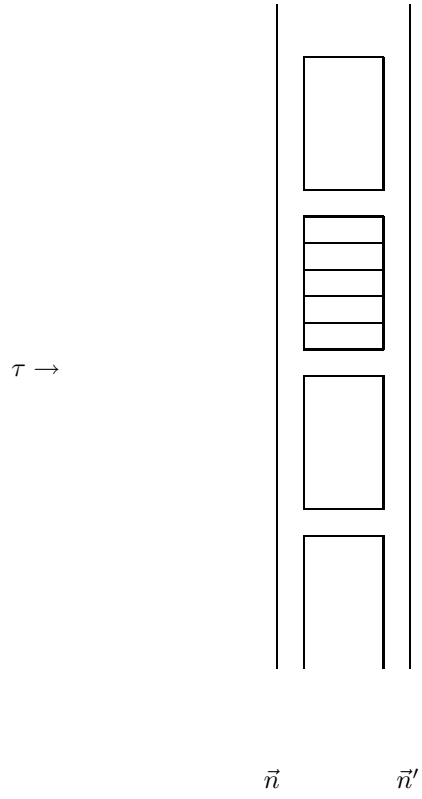
FIG. 1. Dual variable in the Ising model.

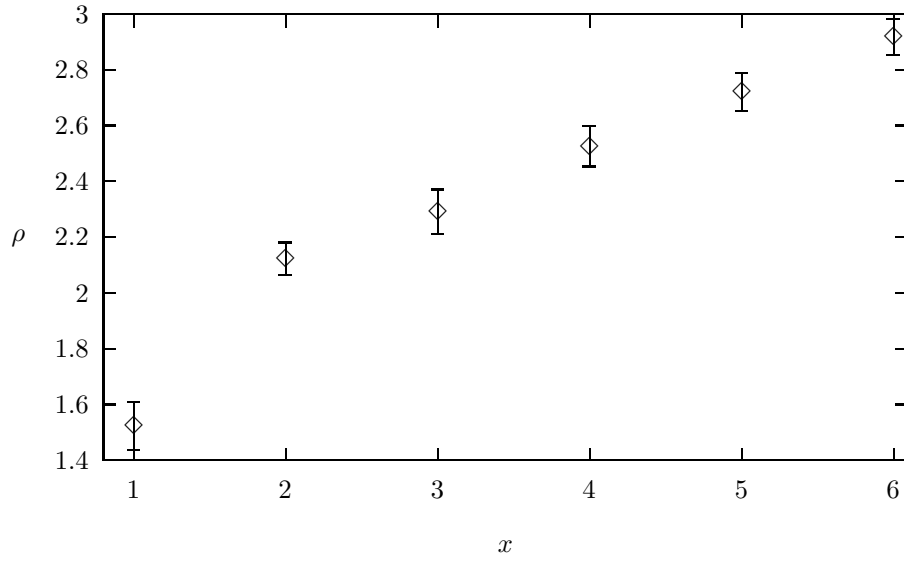FIG. 2. Strong coupling diagram in the finite temperature $SU(2)$ LGT contributing to the effective Ising model.
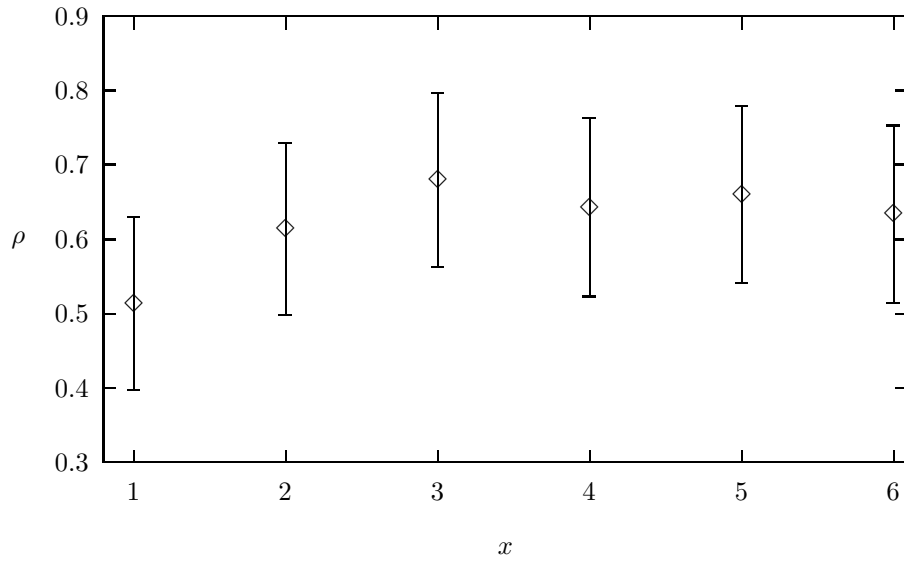
FIG. 3. ρ in the deconfining phase.

FIG. 4. ρ in the confining phase.